# Inductive Posts and Diaphragms of Arbitrary Shape and Number in a Rectangular Waveguide

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***Abstract*—Consider a finite number of posts and/or diaphragms located close to each other in a rectangular waveguide. These are assumed to be perfectly conducting, of arbitrary shape, and uniform in the direction parallel to the narrow side of the waveguide, i.e., of the inductive type. The solution of the problem involves determining the network describing the effect of the posts and diaphragms on the waveguide dominant mode. A moment procedure is devised and applied to a set of test problems. The simplicity and generality of the procedure, together with its excellent performance, as indicated by the results obtained, clearly shows that it is a powerful tool worth using.**

## I. Introduction

A COMMONLY USED building block in microwave filter design consists of a configuration of closely spaced posts or diaphragms in a rectangular waveguide. A knowledge of the characteristics of such a block is required before a synthesis procedure can be applied. In particular, the effect of the block on the dominant waveguide mode must be accurately described, and, from an engineering perspective, descriptions employing networks of lumped elements are preferred. In this paper, the class of inductive posts is considered, and its network description is being sought. (To avoid unnecessary writing, posts, hereafter, are meant to be both of zero thickness, i.e., diaphragms, as well as of finite thickness.) These posts are assumed to be perfectly conducting, of arbitrary cross section, and uniform in the direction parallel to the narrow side of the waveguide. The problem considered is depicted in Fig. 1.

The analysis of posts in a waveguide is an important problem in microwave theory, and a large body of literature exists on it. However, many of the available solutions have shortcomings. The number of posts considered is usually one, and the extension to more than one post is not easy to obtain. Also, the application of these methods is largely restricted to specific posts like circular posts, windows and diaphragms coincident with a waveguide cross section, etc., or those that may be approximated as such. An account of some of these methods can be found in texts [1, ch. 8], [2, chs. 5, 6]. Subsectional moment methods, on the other hand, provide a means by which these solutions can be made more general. For example, a large class of posts, including the ones described above, can be handled without difficulty. As a result, a general computer code which allows a systematic application of the procedure can be written. It is the purpose of this paper to demonstrate such a solution.

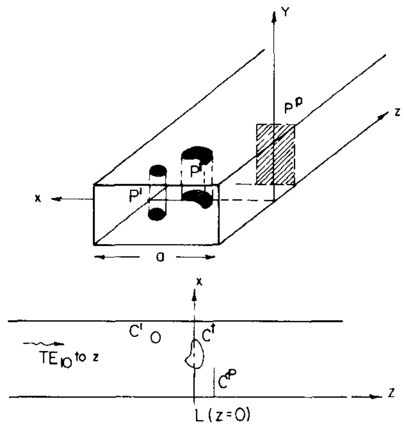


Fig. 1. $p$ inductive posts in a waveguide.

The moment procedure is quite straightforward. First, each post is approximated by a finite number of strips, each of which carries a constant current whose value is to be determined. The Green's function for the $TE_{n0}$ to $z$ modes is then used to express the field produced by these currents. For an incident $TE_{10}$ to $z$ mode traveling along the waveguide axis in either direction, the total tangential electric field must vanish on each post. Satisfaction of this condition along a line in the direction of each strip leads to two systems of equations, one for each excitation. It is worthy of note that in the two cases the matrix is the same, whereas the right-hand side vectors are complex conjugates. The solution of these equations determines the current induced on each post for each excitation. The reflection and transmission coefficients of the waveguide dominant mode are then evaluated using the induced currents to set up the scattering matrix of the posts. The impedance matrix is then determined, and readily realized by a $T$-network of lumped elements, using standard microwave network theory.

This paper is organized as follows. The moment procedure is presented in Section II. The evaluation of the matrix elements is considered in Section III. Section IV presents some of the results obtained by applying the

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procedure to a few selected problems. Final remarks are given in the discussion in Section V.

## II. Basic Formulation

Let $P^1, P^2, \cdots, P^p$ be inductive posts located in a rectangular waveguide whose axis is in the $z$ direction. These posts are assumed to be perfectly conducting, of arbitrary cross section, and uniform along the $y$-axis.

Let a $TE_{10}$ to $z$ mode of unit amplitude be incident on the posts from the left. The incident electric field has only a $y$-component given by

$$E_y^i = \sin\left(\frac{\pi}{a}x\right)e^{-\gamma_1 z} \tag{1}$$

where

$$\gamma_1 = j\frac{2\pi}{\lambda_{10}} = \sqrt{\left(\frac{\pi}{a}\right)^2 - \kappa^2}. \tag{2}$$

Here $\kappa = 2\pi/\lambda = \omega\sqrt{\mu\epsilon}$ is the wavenumber of the medium filling the waveguide, and $\lambda$ is its wavelength. This medium is assumed linear, homogeneous, isotropic, and dissipation free, and is therefore characterized by the real scalar permeability $\mu$ and the real scalar permittivity $\epsilon$. Furthermore, it is assumed that $a < \lambda < 2a$, so that only the dominant mode can propagate in the waveguide.

Since each post is uniform along the $y$-axis, and since the incident field has only a $y$-component electric field that does not vary with $y$, the field scattered from the posts must have only a $y$-component electric field that does not vary with $y$. Thus, the only higher order mode excited are $TE_{n0}$ to $z$ modes, since these are the only modes having only an $E_y$ component that does not vary with $y$. Consequently, the current induced on each post has only a $y$-component that does not vary with $y$. The scattered field can be evaluated in terms of these currents by using the Green's function for $TE_{n0}$ to $z$ modes [1, sec. 5-6]. (The problem is basically a two-dimensional scalar one, and it is assumed, hereafter, that all source and measurement points are located in some plane $y$ = constant within the waveguide.) The scattered field is then given by

$$E_y^s = \sum_{t=1}^{p} \int_{C^t} G(x, z|x', z') J^t(x', z')\, dl' \tag{3}$$

where

$$G(x, z|x', z') = -\frac{j\omega\mu}{a} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right)e^{-\gamma_n|z-z'|}}{\gamma_n} \quad (0 \leqslant x \leqslant a) \tag{4}$$

$$\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \kappa^2} \tag{5}$$

$$dl' = \sqrt{(dx')^2 + (dz')^2}. \tag{6}$$

In (3), $J^t$ is the current induced on the $t$th post, and the integral is taken along each boundary $C^t$ of the $t$th post cross section, $1 \leqslant t \leqslant p$. Since each post is perfectly conducting, we must have

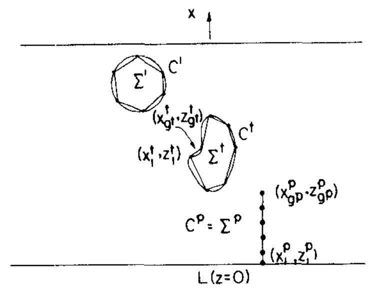


Fig. 2. Each $C^t$, $1 \leqslant t \leqslant p$, approximated by a polygon $\Sigma^t$.

$$E_y^i + E_y^s = 0 \quad \text{on} \quad \bigcup_{t=1}^{p} C^t \tag{7}$$

where $\cup$ denotes the union of sets. This is an integral equation for the induced currents.

An exact solution of (7) can rarely be obtained. An approximate solution can be obtained by replacing each $C^t, 1 \leqslant t \leqslant p$, by a polygon $\Sigma^t = \{S_1^t, S_2^t, \cdots, S_{g^t}^t\}$ (see Fig. 2), each segment of which carries a constant current whose value is to be determined. The integral equation then becomes

$$\sin\left(\frac{\pi}{a}x\right)e^{-\gamma_1 z} + \sum_{t=1}^{p}\sum_{u=1}^{g^t} \alpha_{1u}^t \int_{S_u^t} G(x, z|x', z')\, dl' = 0, \qquad (x, z) \in \bigcup_{t=1}^{p} \Sigma^t. \tag{8}$$

In (8), $\alpha_{1u}^t$ is the unknown constant current on the $u$th segment of $\Sigma^t$, $S_u^t$, along which the integral is taken, $1 \leqslant u \leqslant g^t, 1 \leqslant t \leqslant p$. Satisfying (8) at $\cup_{t=1}^{p} M^t$, where

$$M^t = \left\{\left(x_{v+(1/2)}^t, z_{v+(1/2)}^t\right) : \left(\frac{x_{v+1}^t + x_v^t}{2}, \frac{z_{v+1}^t + z_v^t}{2}\right), \quad 1 \leqslant v \leqslant g^t\right\} \tag{9}$$

and $(x_{g^t+1}^t, z_{g^t+1}^t) = (x_1^t, z_1^t)$ for any post $t$ of finite thickness, we obtain the system of equations

$$\overline{\overline{Z}}\vec{I}_1 = \vec{V}_1 \tag{10}$$

where $\overline{\overline{Z}}$ is a $p$ by $p$ block matrix whose $rs$th block is the $g^r$ by $g^s$ matrix

$$B^{rs} = [\zeta_{vu}^{rs}] = \left[-\int_{S_u^s} G\left(x_{v+(1/2)}^r, z_{v+(1/2)}^r|x', z'\right) dl'\right] \tag{11}$$

and $\vec{I}_1$ and $\vec{V}_1$ are the $p$ segment vectors whose $s$th and $r$th segments are, respectively, the $g^s$ by 1 and $g^r$ by 1 vectors

$$\vec{I}_1^s = [\alpha_{1u}^s] \tag{12}$$

$$\vec{V}_1^r = [\phi_{1v}^r] = \left[\sin\left(\frac{\pi}{a}x_{v+(1/2)}^r\right)e_{v+(1/2)}^{-\gamma_1 z^r}\right]. \tag{13}$$

Solution of the system of equations (10) determines the current induced on each segment.

This approximate solution is a moment solution with pulses used for current expansion and a point-matching procedure [3, sec. 1-5]. The higher order modes excited are evanescent, i.e., decay exponentially with distance from the posts. Thus, at distances sufficiently far from the posts, only the dominant mode propagates in the waveguide. The reflection coefficient of the dominant mode $\Gamma_1$, evaluated at $z=0$, can be determined from (4) and (8) as

$$\Gamma_1 = -\frac{j\omega\mu}{a\gamma_1}\sum_{t=1}^{p}\sum_{u=1}^{g^t}\alpha_{1u}^t\int_{S_u^t}\sin\left(\frac{\pi}{a}x'\right)e^{-\gamma_1 z'}\,dl'. \quad (14)$$

The transmission coefficient of the dominant mode, evaluated at $z=0$, is then given by

$$T_1 = 1-\frac{j\omega\mu}{a\gamma_1}\sum_{t=1}^{p}\sum_{u=1}^{g^t}\alpha_{1u}^t\int_{S_u^t}\sin\left(\frac{\pi}{a}x'\right)e^{\gamma_1 z'}\,dl'. \quad (15)$$

The choice of the reference plane $L$ at $z=0$ is only a matter of convenience.

In view of the evanescing nature of the higher order modes, the posts can be described by their effect on the dominant mode. To do this, let a $TE_{10}$ to $z$ mode of unit amplitude be incident on the posts from the right. This mode has only a $y$-component electric field, which is now given by

$$E_y^i = \sin\left(\frac{\pi}{a}x\right)e^{\gamma_1 z}. \quad (16)$$

Let each $C^t, 1 \leqslant t \leqslant p$, be approximated in exactly the same way as done previously, i.e., by the same polygon $\Sigma^t = \{S_1^t, S_2^t, \cdots, S_{g^t}^t\}$. Each segment is assumed to carry an unknown constant current $\alpha_{2u}^t$. Using the Green's function $G(x,z|x',z')$ to express the scattered field $E_y^s$, then point matching $E_y^i + E_y^s$ to zero at $\cup_{t=1}^{p} M^t$, where $M^t$ is the point set defined by (10), we obtain the system of equations

$$\overline{\overline{Z}}\vec{I}_2 = \vec{V}_2. \quad (17)$$

Here $\overline{\overline{Z}}$ is the $p$ by $p$ block matrix given by (11), and $\vec{I}_2$ and $\vec{V}_2$ are the $p$ segment vectors whose $s$th and $r$th segments are the $g^s$ by 1 and $g^r$ by 1 vectors

$$\vec{I}_2^s = [\alpha_{2u}^s] \quad (18)$$

$$\vec{V}_2^r = [\phi_{2v}^r] = \left[\sin\left(\frac{\pi}{a}x_{v+(1/2)}^r\right)e^{\gamma_1 z_{v+(1/2)}^r}\right] \quad (19)$$

respectively. Solution of the system of equations (17) determines the current induced on each segment. The reflection and transmission coefficients of the dominant mode, evaluated at $z=0$, in this case are given by

$$\Gamma_2 = -\frac{j\omega\mu}{a\gamma_1}\sum_{t=1}^{p}\sum_{u=1}^{g^t}\alpha_{2u}^t\int_{S_u^t}\sin\left(\frac{\pi}{a}x'\right)e^{\gamma_1 z'}\,dl' \quad (20)$$

$$T_2 = 1-\frac{j\omega\mu}{a\gamma_1}\sum_{t=1}^{p}\sum_{u=1}^{g^t}\alpha_{2u}^t\int_{S_u^t}\sin\left(\frac{\pi}{a}x'\right)e^{-\gamma_1 z'}\,dl'. \quad (21)$$

The effect of the posts on the dominant mode can then be described by the scattering matrix [4, sec. 5-14]

$$S = \begin{bmatrix}\Gamma_1 & T_2\\ T_1 & \Gamma_2\end{bmatrix}. \quad (22)$$

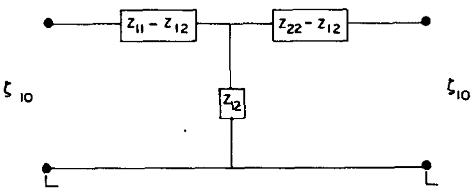


Fig. 3. A two-port network corresponding to the matrix $\overline{\overline{Z}}$ in (27).

For a reciprocal medium, $T_1$ and $T_2$ are necessarily equal. In the Galerkin case with pulses used for both current expansion and testing, it is easy to show that the moment matrix $\overline{\overline{Z}}$ is symmetric, and $T_1$ and $T_2$ are given by

$$T_1 = 1-\frac{j\omega\mu}{a\gamma_1}\vec{I}_1'^T\vec{V}_2' \quad (23)$$

$$T_2 = 1-\frac{j\omega\mu}{a\gamma_1}\vec{I}_2'^T\vec{V}_1' \quad (24)$$

where the superscript $T$ denotes vector transpose. Consequently, $T_1$ and $T_2$ are equal. This need not be the case when point matching is used to satisfy the boundary conditions. However, the modulus of the difference in transmission coefficients is small so that an average transmission coefficient may conveniently be defined. To proceed further, define the average transmission coefficient

$$T_{av} = \frac{T_1+T_2}{2}. \quad (25)$$

The scattering matrix of the posts then becomes

$$S = \begin{bmatrix}\Gamma_1 & T_{av}\\ T_{av} & \Gamma_2\end{bmatrix}. \quad (26)$$

The posts impedance matrix is given by

$$Z = \zeta_{10}(U+S)(U-S)^{-1} = \begin{bmatrix} z_{11} & z_{12}\\ z_{12} & z_{22}\end{bmatrix} \quad (27)$$

where $\zeta_{10} = j\omega\mu/\gamma_1$ is the characteristic impedance of the $TE_{10}$ to $z$ mode, and $U$ is the identity matrix. For a lossless medium and perfectly conducting posts, the elements of $Z$ are pure imaginary [4, sec. 5-12]. However, due to the approximations involved in the numerical solution, the imaginary nature of $Z$ need not be preserved ($S$ need not be unitary). The real part of $Z$ is of small norm, nevertheless, so that it is reasonable to consider only the imaginary part of $Z$. The equivalent network corresponding to (27) is shown in Fig. 3.

## III. Evaluation of Matrix Elements

The construction of the moment matrix $\overline{\overline{Z}}$ in (10) and (17), constitutes a large portion of the work involved in the moment solution. An efficient evaluation of the matrix elements is therefore necessary for the success of the solution. A typical element of $\overline{\overline{Z}}$ is given by the integral

$$-\int_{S_u^s} G(x_{v'}, z_{v'}|x',z')\,dl'$$

where

$$(x_{v'}, z_{v'}) = (x_{v+(1/2)}^r, z_{v+(1/2)}^r).$$

The first step is to express the dynamic Green's function $G(x_{v'}, z_{v'}|x',z')$ in terms of the corresponding static

TABLE I
THE PARAMETERS OF AN EIGHT-POINT GAUSS-RADAU QUADRATURE RULE

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $p_i$ | 0.0 | 0.06412993 | 0.20414991 | 0.39535039 | $1-p_4$ | $1-p_3$ | $1-p_2$ | $1-p_1$ |
| $q_i$ | 0.03571428 | 0.21070422 | 0.34112270 | 0.41245880 | $q_4$ | $q_3$ | $q_2$ | $q_1$ |

Green's function, whose analytic form is known, plus correction terms. The integration is then carried out numerically.

Put

$$\gamma_1 = \sqrt{\left(\frac{\pi}{a}\right)^2 - \kappa^2} = j\frac{\pi}{a}\sqrt{\left(\frac{2a}{\lambda}\right)^2 - 1} = j\frac{\pi}{a}\beta_1 \tag{28}$$

$$\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \kappa^2} = \frac{\pi}{a}\sqrt{n^2 - \left(\frac{2a}{\lambda}\right)^2} = \frac{\pi}{a}\beta_n \qquad (n \geq 2). \tag{29}$$

Then

$$-G(x_{v'}, z_{v'}|x', z') = \frac{\omega\mu}{\pi\beta_1}\sin\left(\frac{\pi}{a}x_{v'}\right)\sin\left(\frac{\pi}{a}x'\right) \cdot e^{-j(\pi/a)|z_{v'}-z'|\beta_1} + \frac{j\omega\mu}{\pi}G' \tag{30}$$

where

$$G' = \sum_{n=2}^{\infty} \frac{\sin\left(\frac{n\pi}{a}x_{v'}\right)\sin\left(\frac{n\pi}{a}x'\right)}{\beta_n} e^{-(\pi/a)|z_{v'}-z'|\beta_n}$$
$$= -\sin\left(\frac{\pi}{a}x_{v'}\right)\sin\left(\frac{\pi}{a}x'\right)e^{-(\pi/a)|z_{v'}-z'|} - \frac{\pi}{\mu}G^{st} + G''. \tag{31}$$

In (31), $G^{st}$ is the static Green's function

$$G^{st}(x_{v'}, z_{v'}|x', z') = \frac{-\mu}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{a}x_{v'}\right)\sin\left(\frac{n\pi}{a}x'\right) \cdot e^{-(n\pi/a)|z_{v'}-z'|} \tag{32}$$

which can be obtained by setting $\kappa$ equal to zero and dropping the $j\omega$ factor in (4), whereas $G''$ is the correction series

$$G'' = \sum_{n=2}^{\infty}\sin\left(\frac{n\pi}{a}x_{v'}\right)\sin\left(\frac{n\pi}{a}x'\right) \cdot \left(\frac{e^{-(\pi/a)|z_{v'}-z'|\beta_n}}{\beta_n} - \frac{e^{-(n\pi/a)|z_{v'}-z'|}}{n}\right). \tag{33}$$

The series in (32) is readily summed (see Appendix A) to give

$$-\frac{\pi}{\mu}G^{st} = \frac{1}{2}\mathrm{Re}\left[\log\left(\frac{1-e^{-(\pi/a)(j|x_{v'}+x'|+|z_{v'}-z'|)}}{1-e^{-(\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)}}\right)\right] \tag{34}$$

where log is meant to be the natural logarithm. The correction series $G''$ is dominated by an exponentially convergent series of positive monotonically decreasing terms (see Appendix B), and may therefore be summed directly at a very modest cost. Combining (30), (31), and (34), we obtain a new form of $G$

$$-G(x_{v'}, z_{v'}|x', z') = \frac{\omega\mu}{\pi\beta_1}\sin\left(\frac{\pi}{a}x_{v'}\right)\sin\left(\frac{\pi}{a}x'\right) \cdot e^{-(j\pi/a)|z_{v'}-z'|\beta_1}$$
$$+\frac{j\omega\mu}{\pi}\left(\sin\left(\frac{\pi}{a}x_{v'}\right)\sin\left(\frac{\pi}{a}x'\right)e^{-(\pi/a)|z_{v'}-z'|} + \frac{1}{2}\mathrm{Re}\left[\log\left(\frac{1-e^{-(\pi/a)(j|x_{v'}+x'|+|z_{v'}-z'|)}}{1-e^{-(\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)}}\right)\right] + G''\right). \tag{35}$$

The integration is now carried out numerically with $G$ given by (35). Apart from $j\omega G^{st}$, the integration of the terms composing $G$ causes no difficulty, and any quadrature rule can be used. Thus, each integral is computed as

$$\int_{S_u^s} T\,dl' \sim \frac{l_u^s}{2}\sum_{i=1}^{N} q_i T(x_{v'}, z_{v'}|(1-p_i)x_u^s + P_i x_{u+1}^s \cdot (1-p_i)z_u^s + p_i z_{u+1}^s) \tag{36}$$

where

$$l_u^s = \sqrt{(x_{u+1}^s - x_u^s)^2 + (z_{u+1}^s - z_u^s)^2}. \tag{37}$$

In (36), $N$ is the order of the rule, the $q_i$ are its coefficients, and the $p_i$ determine the location of its abscissas. $T$ stands for any term in (35) except $j\omega G^{st}$. Table I shows the $q_i$ and $p_i$ of an eight-point Gauss-Radau rule [5].

For any off-diagonal element of $\bar{\bar{Z}}$, the $j\omega G^{st}$ term can be integrated via (36). When evaluating the diagonals of $\bar{\bar{Z}}$, $j\omega G^{st}$ offers a logarithmic singularity at $(x_{v'}, z_{v'})$ that requires particular attention. In a very small neighborhood $((x_{v'} - \delta_x, z_{v'} - \delta_z), (x_{v'} + \delta_x, z_{v'} + \delta_z))$, $\delta_x, \delta_z > 0$, about $(x_{v'}, z_{v'})$ the following approximation is valid:

$$\mathrm{Re}\left[\log\left(1 - e^{-(\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)}\right)\right] \sim \mathrm{Re}\left[\log\left(\frac{\pi}{a}(j|x_{v'}-x'| + |z_{v'}-z'|)\right)\right] = \log\left(\frac{\pi}{a}\rho\right) \tag{38}$$

where

$$\rho = \sqrt{(x_{v'} - x')^2 + (z_{v'} - z')^2}. \tag{39}$$

The integral of the singular function $j\omega G^{st}$ can then be written as

$$\int_{S_u^s} j\omega G^{st}\,dl' = \frac{-1}{2}\int_{S_u^s}\log\left(\frac{\pi}{a}\rho\right)dl' + \int_{S_u^s}\left(j\omega G^{st} + \frac{1}{2}\log\left(\frac{\pi}{a}\rho\right)\right)dl'$$
$$= -\frac{l_u^s}{2}\left(\log\left(\frac{\pi}{a}\frac{l_u^s}{2}\right) - 1\right) + \int_{S_u^s}\left(j\omega G^{st} + \frac{1}{2}\log\left(\frac{\pi}{a}\rho\right)\right)dl'. \tag{40}$$

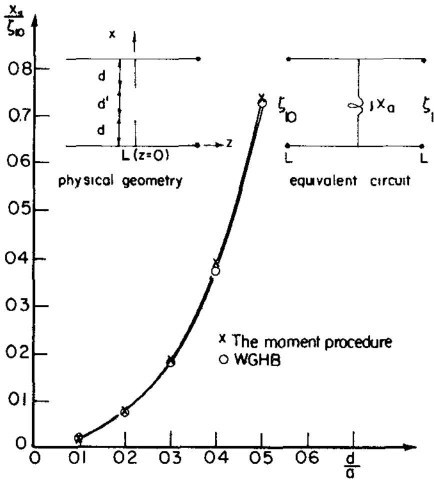


Fig. 4. The reactance of the symmetrical thin window ($a/\lambda = 0.8$).

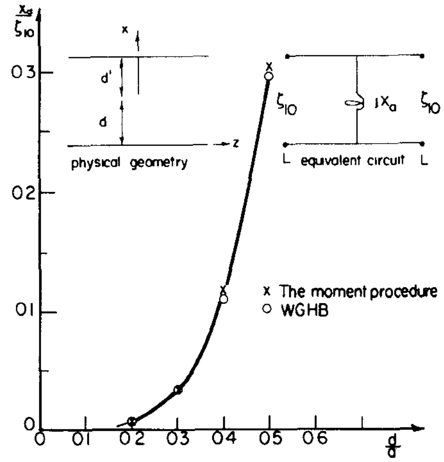


Fig. 6. The reactance of the asymmetrical thin window. ($a/\lambda = 0.8$).

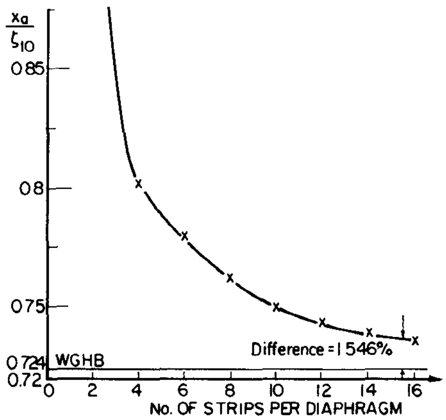


Fig. 5. The convergence behavior of the moment procedure for the symmetrical thin window ($a/\lambda = 0.8$, $d/a = 0.5$).

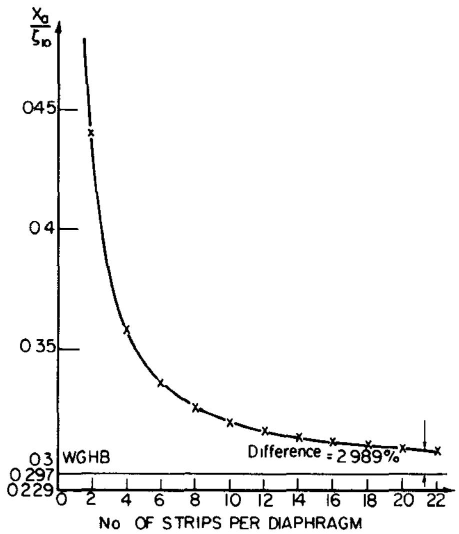


Fig. 7. The convergence behavior of the moment procedure for the asymmetrical thin window ($a/\lambda = 0.8$, $d/a = 0.5$).

The integral on the right-hand side of (40) has no singularity at $(x_{v'}, z_{v'})$, and can be computed using (36).

## IV. Testing the Moment Solution

The solution procedure presented has been translated into a computer program. To test the moment solution, the computer program is run for a few selected problems. Some of these problems are of practical importance, and have been treated in the literature. In particular, the problems of the symmetrical thin window, the asymmetrical thin window, the circular post, and the triple circular post are considered.

The elements of the scattering matrix and the reactances of the equivalent $T$-network are basically the parameters to be computed. In each case, the resulting equivalent network is compared to that given in the *Waveguide Handbook* (WGHB) [6]. The equivalent network of the triple circular post is tested against measured data. A complete assessment of the performance of the moment procedure should also consider the (Frobenius) norm of the real part of $Z$ and the modulus of the difference in transmission coefficients. These two numbers are computed in all program runs, and are usually found to be $O(10^{-8})$ or better.

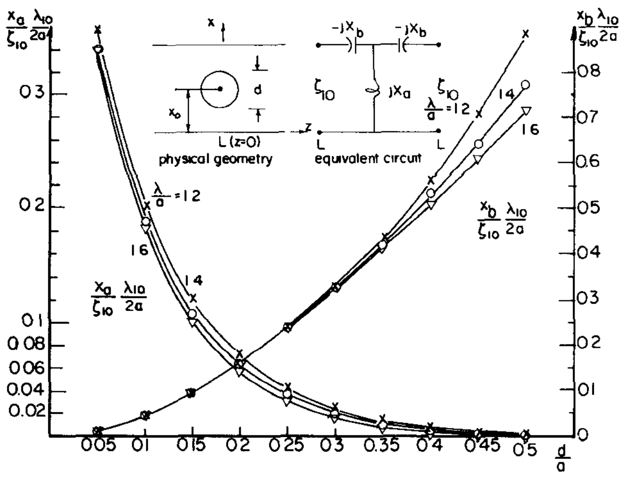


Fig. 8. Network reactances of a centered circular post. (The number of strips approximating the post is 24.)

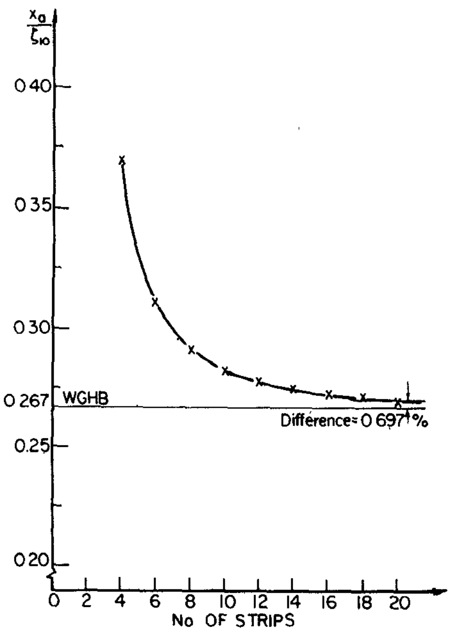


Fig. 9. The convergence behavior of the moment procedure for the centered circular post ($\lambda/a = 1.2$, $d/a = 0.1$).

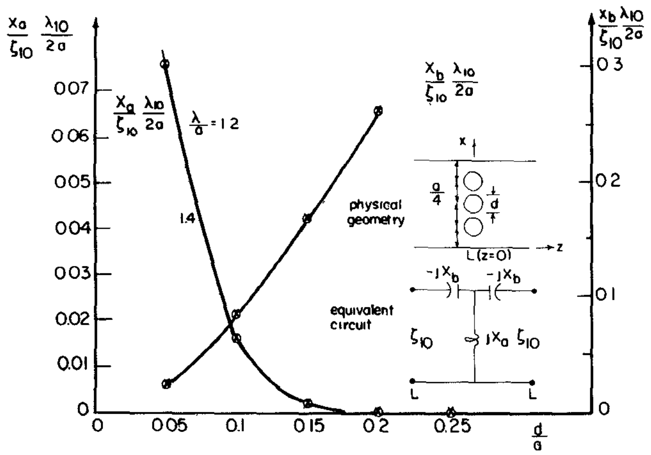


Fig. 10. Network reactances of a symmetrical triple circular post. (The number of strips approximating each post is 16.)

Occasionally, the latter gets to be $O(10^{-5})$. The problems considered, and some of the obtained results are shown in Figs. 4–11.

Some general remarks can be drawn from the results obtained.

In all the cases, the convergence for the inductive reactance $X_a$ is monotonic and from above, as can be readily seen from the set of figures. This can be shown for the Galerkin procedure, and is true as an approximation for the present procedure. Thus, the true value of $X_a$ can be considered the greatest lower bound (infimum) of its computed values, and the difference margin is actually less than that indicated in the figures. The computed reactances, nevertheless, agree well with the WGHB data, with only a few strips needed to approximate even large posts.

Perhaps the most interesting observation can be drawn by examining Fig. 8 for the centered circular post. For large posts ($d/a > 0.25$), $X_b\lambda_{10}/2a\zeta_{10}$ is no longer frequency independent, as is the case with small posts ($d/a \leqslant 0.25$), but rather branches out. Such a behavior has been confirmed by Leviatan *et al.* [7]. Figs. 10–11 for the symmetrical triple circular post display yet another set of almost frequency-independent characteristics. This is not surprising, however, since this configuration cancels out the first six higher order modes [2, sec. 5-1.3].

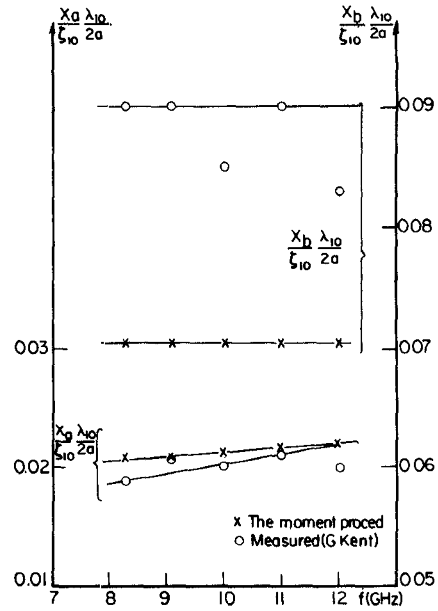


Fig. 11. Measured and computed reactances of the symmetrical triple circular post (diameter = 0.089″) in a WR-90 waveguide. (16 strips are used to approximate each post.)

## V. Discussion

In this paper, a subsectional point-matching moment procedure has been applied to multiple posts of the inductive type located in a rectangular waveguide. These posts are metallic, of arbitrary cross section, and uniform in the direction parallel to the narrow side of the waveguide.

In the procedure, the current induced on each post is expanded in terms of pulses. Functions other than pulses could be used, but then the integrals in the moment matrix would be more complicated. Experience has shown that a substantial part of the total work involved in the solution is done in constructing the moment matrix. A pulse representation of the current, with point matching to satisfy the boundary conditions, is used so as to render the procedure economical. However, this has been very successful, as is evident by the performance of the procedure.

The moment procedure can also be applied, with due changes, to other classes of posts. For instance, only few changes are needed so that the procedure may apply to the dual class of capacitive posts, i.e., posts that are metallic, of arbitrary cross section, and uniform in the direction parallel to the broad side of the waveguide. The class of dielectric posts in the inductive, or capacitive, position may also be handled using the moment procedure, but major revisions will have to be made. These cases are currently being considered.

## Appendix A

Consider the static Green's function

$$
\begin{aligned}
-\frac{\pi}{\mu} G^{st}(x_{v'}, z_{v'}|x', z')
&= \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x_{v'}}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) e^{-(n\pi/a)|z_{v'}-z'|} \\
&= \frac{1}{2}\left( \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\pi}{a}|x_{v'}-x'|\right) e^{-(n\pi/a)|z_{v'}-z'|} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\pi}{a}|x_{v'}+x'|\right) e^{-(n\pi/a)|z_{v'}-z'|} \right) \\
&= \frac{1}{2} \operatorname{Re}\left[ \sum_{n=1}^{\infty} \frac{1}{n} e^{-(n\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \frac{1}{n} e^{-(n\pi/a)(j|x_{v'}+x'|+|z_{v'}-z'|)} \right]. \qquad \text{(A1)}
\end{aligned}
$$

Since

$$\frac{1}{1-\eta} = \sum_{n=0}^{\infty} \eta^n \qquad (\eta \in \mathbb{C}, |\eta| < 1) \qquad \text{(A2)}$$

and the series in (A2) is uniformly convergent for $0 \leqslant |\eta| \leqslant |\xi| < 1$, a term-by-term integration can be carried out [8, sec. 5-4], giving

$$
\begin{aligned}
\int_0^{\xi} \frac{1}{1-\eta}\, d\eta &= \sum_{n=0}^{\infty} \int_0^{\xi} \eta^n\, d\eta \\
-\log(1-\xi) &= \sum_{n=0}^{\infty} \frac{\xi^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{\xi^n}{n}. \qquad \text{(A3)}
\end{aligned}
$$

Putting $\xi$ equal to $e^{-(\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)}$ and $e^{-(\pi/a)(j|x_{v'}+x'|+|z_{v'}-z'|)}$ in (A3), then using the results in (A1), the static Green's function reads

$$
\begin{aligned}
-\frac{\pi}{\mu} G^{st}(x_{v'}, z_{v'}|x', z')
&= \frac{1}{2} \operatorname{Re}\left[ \log\left( \frac{1 - e^{-(\pi/a)(j|x_{v'}+x'|+|z_{v'}-z'|)}}{1 - e^{-(\pi/a)(j|x_{v'}-x'|+|z_{v'}-z'|)}} \right) \right] \qquad \text{(A4)}
\end{aligned}
$$

as asserted.

## Appendix B

Consider the correction series

$$G'' = \sum_{n=2}^{\infty} a_n \sin\left(\frac{n\pi}{a} x_{v'}\right) \sin\left(\frac{n\pi}{a} x'\right) \qquad \text{(B1)}$$

where

$$a_n = \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_n}}{\beta_n} - \frac{e^{-(n\pi/a)|z_{v'}-z'|}}{n} \qquad \text{(B2)}$$

$$\beta_n = \sqrt{n^2 - \left(\frac{2a}{\lambda}\right)^2} \qquad \left(1 < \frac{2a}{\lambda} < 2\right). \qquad \text{(B3)}$$

Clearly

$$|G''| \leqslant \sum_{n=2}^{\infty} |a_n|. \qquad \text{(B4)}$$

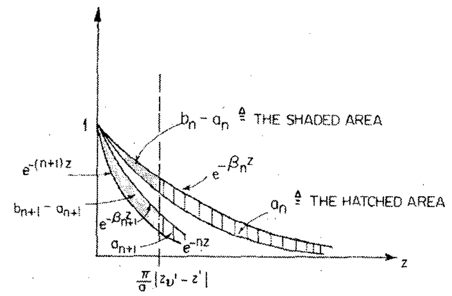


Fig. 12. Pictorial illustration of the procedure in the Appendix.

Put

$$a(t) = \frac{e^{-(\pi/a)|z_{v'}-z'|\beta(t)}}{\beta(t)} - \frac{e^{-(t\pi/a)|z_{v'}-z'|}}{t} \qquad \text{(B5)}$$

$$\beta(t) = \sqrt{t^2 - \left(\frac{2a}{\lambda}\right)^2} \qquad (t \geqslant 2) \qquad \text{(B6)}$$

$$f(z,t) = e^{-\beta(t)} - e^{-tz}. \qquad \text{(B7)}$$

Then

$$a(t) = \int_{(\pi/a)|z_{v'}-z'|}^{\infty} f(z,t)\, dz. \qquad \text{(B8)}$$

Since $t > \beta(t)$ for all $t \geqslant 2$, then $a(t) > 0$ for all $t \geqslant 2$, and certainly so is $a_n$, $n = 2, 3, \cdots$.

That $\{a_n\}$ is a monotonically decreasing sequence follows from its positiveness. Since $f$ and $\partial/\partial t f$ are continuous for $t \geqslant 2$ and for all $z \geqslant (\pi/a)|z_{v'} - z'|$, and the integrals

$$\int_{(\pi/a)|z_{v'}-z'|}^{\infty} f\, dz \text{ and } \int_{(\pi/a)|z_{v'}-z'|}^{\infty} \left(\frac{\partial}{\partial t} f\right) dz$$

converge uniformly for all $t \geqslant 2$, then [9, sec. 7-5]

$$
\begin{aligned}
\frac{d}{dt} a(t) &= \int_{(\pi/a)|z_{v'}-z'|}^{\infty} \left(\frac{\partial}{\partial t} f\right) dz \\
&= -\int_{(\pi/a)|z_{v'}-z'|}^{\infty} zt\left(\int_z^{\infty} f(z,t)\, dz\right) dz \qquad \text{(B9)}
\end{aligned}
$$

with the help of (B5) and (B8). Consequently, $d/dt\, a(t) < 0$. Thus, $a(t)$ is a monotonically decreasing function for all $t \geqslant 2$, and certainly so is $a_n$, $n = 2, 3, \cdots$.

Thus, the correction series $G''$ is dominated by $\Sigma_{n=2}^{\infty} a_n$, a series of positive monotonically decreasing terms. Since

$$\beta_{n+1} > n, \qquad n = 2, 3, \cdots \qquad \text{(B10)}$$

then

$$a_n < \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_n}}{\beta_n} - \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_{n+1}}}{\beta_{n+1}}, \qquad n = 2, 3, \cdots. \qquad \text{(B11)}$$

Consequently

$$\sum_{n=2}^{N} a_n < \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_2}}{\beta_2} - \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_{N+1}}}{\beta_{N+1}} \qquad \text{(B12)}$$

whence

$$|G''| \leqslant \sum_{n=2}^{\infty} a_n < \frac{e^{-(\pi/a)|z_{v'}-z'|\beta_2}}{\beta_2} \tag{B13}$$

as

$$\frac{e^{-(\pi/a)|z_{v'}-z'|\beta_{N+1}}}{\beta_{N+1}} \to 0 \text{ as } N \to \infty.$$

The second inequality in (B13) can also be deduced from Fig. 12. It follows from (B12) that $G''$ converges exponentially. Furthermore, since

$$a_n \leqslant b_n = \frac{1}{\beta_n} - \frac{1}{n}, \qquad n = 2, 3, \cdots \tag{B14}$$

$$|G''| \leqslant \sum_{n=2}^{\infty} a_n \leqslant \sum_{n=2} b_n < \frac{1}{\beta_2} \tag{B15}$$

which can be proved using similar procedure, and is also evident from Fig. 12, $G''$ does converge uniformly for all $(x', z')$.

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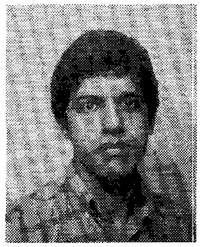

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# Composite Coupler Design

THOMAS C. CHOINSKI

*Abstract* —**Unequal power splitters and combiners are generally limited by the line widths which can be practically synthesized in a given transmission medium. This practical limitation on the ratio of unequal power division can be extended by incorporating the same types of couplers into a composite design.**

**The general composite design approach outlined in this paper uses three couplers (three terminal couplers) to generate a new three-terminal circuit. The design equations are derived for the composite approach and summarized in graphic form.**

**The feasibility of the composite design approach is demonstrated by the construction of a 5.76-dB differential coupler using internally series-terminated Wilkinson couplers. The circuit was designed, analyzed via computer, and finally built and tested. The results from the composite design are compared to that of a single Wilkinson coupler design.**

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